

# UNITS IN $F_2D_{2p}$

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**ABSTRACT.** Let  $p$  be an odd prime,  $D_{2p}$  be the dihedral group of order  $2p$ , and  $F_2$  be the finite field with two elements. If  $*$  denotes the canonical involution of the group algebra  $F_2D_{2p}$ , then bicyclic units are unitary units. In this note, we investigate the structure of the group  $\mathcal{B}(F_2D_{2p})$ , generated by the bicyclic units of the group algebra  $F_2D_{2p}$ . Further, we obtain the structure of the unit group  $\mathcal{U}(F_2D_{2p})$  and the unitary subgroup  $\mathcal{U}_*(F_2D_{2p})$ , and we prove that both  $\mathcal{B}(F_2D_{2p})$  and  $\mathcal{U}_*(F_2D_{2p})$  are normal subgroups of  $\mathcal{U}(F_2D_{2p})$ .

## INTRODUCTION

Let  $FG$  be the group algebra of the group  $G$  over the field  $F$  and  $\mathcal{U}(FG)$  denotes its unit group. The anti-automorphism  $g \mapsto g^{-1}$  of  $G$  can be extended linearly to an anti-automorphism  $a \mapsto a^*$  of the group algebra  $FG$  known as canonical involution of  $FG$ . Let  $\mathcal{U}_*(FG)$  be the unitary subgroup consisting of the elements of  $\mathcal{U}(FG)$  that are inverted by canonical involution  $*$ . These elements are called unitary units in  $FG$ . If  $F$  is a finite field of characteristic 2, then  $\mathcal{U}_*(FG)$  coincides with  $V_*(FG)$ ; otherwise, it coincides with  $V_*(FG) \times \langle -1 \rangle$ . Here  $V_*(FG)$  denotes the set of all unitary units in the normalized

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unit group  $V(FG)$ . Interest in the group  $U_*(FG)$  arose in algebraic topology and unitary  $K$ -theory [6].

We are interested in the structure of the unit group  $\mathcal{U}(F_2D_{2p})$  and the unitary subgroup  $\mathcal{U}_*(F_2D_{2p})$ . For a finite abelian  $p$ -group  $G$  and the field  $F$  with  $p$  elements, R. Sandling [7] gave the structure of  $V(FG)$  and the structure of  $V_*(FG)$  was obtained by A.A.Bovdi and A.A.Sakach in [2] for a finite field  $F$  of characteristic  $p$ . For a field  $F$  with two elements and a 2-group  $G$  upto order 16, R. Sandling [8] gave the presentation for  $V(FG)$ . Later on, A.Bovdi and L. Erdei [1] described the structure of the unitary subgroup  $V_*(F_2G)$ , where  $G$  is a nonabelian group of order 8 and 16. In [4] V. Bovdi and A. L. Rosa computed the order of the unitary subgroup of the group of units when  $G$  is either an extraspecial 2-group or the central product of such a group with a cyclic group of order 4, and  $F$  is a finite field of characteristic 2. In the same paper, they computed the order of the unitary subgroup  $V_*(FG)$ , where  $G$  is a 2-group with an abelian subgroup  $A$  of index 2 and an element  $b$  such that  $b$  inverts every element in  $A$  and the order of  $b$  is 2 or 4. V. Bovdi and T.Rozgonyi in [3] described the structure of  $V_*(F_2G)$ , where the order of  $b$  is 4.

For a dihedral group  $G$  of order 6 and 10 and an arbitrary finite field  $F$ , the structure of the unit group  $\mathcal{U}(FG)$  is described in [9] and [5]. Here we give the structure of the unit group  $\mathcal{U}(F_2D_{2p})$  and the group  $\mathcal{U}_*(F_2D_{2p})$ . The bicyclic units of  $F_2D_{2p}$  play an important role in finding the structure of unit group. We also study the structure of the group,  $\mathcal{B}(F_2D_{2p})$ , generated by bicyclic units of  $F_2D_{2p}$ .

For an element  $g \in G$  of order  $n$ , write  $\widehat{g} = 1 + g + g^2 + \cdots + g^{n-1}$ .

If  $g, h \in G$ ,  $o(g) < \infty$ , then

$$u_{g,h} = 1 + (g - 1)h\widehat{g}$$

has an inverse  $u_{g,h}^{-1} = 1 - (g - 1)h\widehat{g}$ . Moreover,  $u_{g,h} = 1$  if and only if  $h$  is in the normalizer of  $\langle g \rangle$ . The element  $u_{g,h}$  is known as a bicyclic unit of the group algebra  $FG$  and the group generated by them is denoted by  $\mathcal{B}(FG)$ . Observe that all nontrivial bicyclic units of the group algebra  $F_2D_{2p}$  are unitary units with respect to canonical involution.

Let  $F_q$  be a finite field with  $q$  elements and  $n$  be a positive integer coprime with  $q$ . If order of  $q \bmod n$  is  $d$ , then the set  $\{a_0, a_0q, \dots, a_0q^{d-1}\}$  of elements of  $\mathbb{Z}_n$  is said to be  $q$ -cycle modulo  $n$ . Further, if  $\alpha$  is a primitive  $n$ -th root of unity, then the polynomial

$$f_{a_0}(x) = (x - \alpha^{a_0})(x - \alpha^{a_0q}) \cdots (x - \alpha^{a_0q^{d-1}}),$$

is an irreducible factor of  $\phi_n(x)$  over  $F_q$  is of degree  $d$ . Hence, the number of irreducible factors of  $\phi_n(x)$  over  $F_q$  is  $\frac{\phi(n)}{d}$ . Since  $F_2$  is a field with 2 elements and  $D_{2p}$  is the dihedral group of order  $2p$ , it follows that if order of 2  $\bmod p$  is  $d$ , then the number of irreducible factors of the cyclotomic polynomial  $\phi_p(x)$  over  $F_2$  is  $\frac{\phi(p)}{d}$  and each irreducible factor is of degree  $d$ .

#### UNIT GROUP OF $F_2D_{2p}$

**Theorem 1.** *Let  $G$  be the dihedral group*

$$D_{2p} = \langle a, b \mid a^p = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

Suppose  $V = \langle 1 + \widehat{D_{2p}} \rangle$ , where  $\widehat{D_{2p}}$  denotes the sum of all elements of  $D_{2p}$ . Then

$$\mathcal{U}(F_2 D_{2p})/V \cong \begin{cases} \underbrace{GL_2(F_{2^{\frac{d}{2}}}) \times GL_2(F_{2^{\frac{d}{2}}}) \cdots \times GL_2(F_{2^{\frac{d}{2}}})}_{\frac{\phi(p)}{d} \text{ copies}}, & \text{if } d \text{ is even} \\ \underbrace{GL_2(F_{2^d}) \times GL_2(F_{2^d}) \cdots \times GL_2(F_{2^d})}_{\frac{\phi(p)}{2d} \text{ copies}}, & \text{if } d \text{ is odd.} \end{cases}$$

$$\text{and hence } |\mathcal{U}(F_2 D_{2p})| = \begin{cases} 2((2^d - 1)(2^d - 2^{\frac{d}{2}}))^{\frac{\phi(p)}{d}}, & \text{if } d \text{ is even} \\ 2((2^{2d} - 1)(2^{2d} - 2^d))^{\frac{\phi(p)}{2d}}, & \text{if } d \text{ is odd} \end{cases}$$

We need the following lemmas:

**Lemma 2.** *Let  $p$  be an odd prime such that order of 2 mod  $p$  is  $d$ . If  $\zeta$  is a primitive  $p$ -th root of unity, then  $\zeta$  and  $\zeta^{-1}$  are the roots of the same irreducible factor of  $\phi_p(x)$  over  $F_2$  if and only if  $d$  is even.*

*Proof.* Assume that  $\zeta$  and  $\zeta^{-1}$  are the roots of the same irreducible factor of  $\phi_p(x)$  over  $F_2$ . It follows that  $-1$  and  $1$  are in same 2-cycle mod  $p$  and so there exist some  $t < d$  such that  $2^t \equiv -1 \pmod{p}$ . Hence order of  $2^t \pmod{p}$  is 2. Further, since  $2^d \equiv 1 \pmod{p}$ , it implies that  $(2^t)^d \equiv 1 \pmod{p}$ . Hence  $2|d$ .

Conversely, let  $d$  be even, say  $d = 2t$ . Then  $2^t \equiv -1 \pmod{p}$  and hence  $-1$  and  $1$  are in same 2-cycle mod  $p$ . The result follows.  $\square$

**Lemma 3.** *Let  $\zeta$  be a primitive  $p$ -th root of unity. If order of 2 mod  $p$  is  $d$ , then  $[F_2(\zeta + \zeta^{-1}) : F_2] = \frac{d}{2}$ , if  $d$  is even and  $[F_2(\zeta + \zeta^{-1}) : F_2] = d$ , if  $d$  is odd and in this case  $F_2(\zeta + \zeta^{-1}) = F_2(\zeta)$ .*

*Proof.* We claim that  $[F_2(\zeta) : F_2(\zeta + \zeta^{-1})] = 1$  or  $2$ . If  $[F_2(\zeta) : F_2(\zeta + \zeta^{-1})] = s > 2$ , then the degree of the minimal polynomial of  $(\zeta + \zeta^{-1})$  over  $F_2$  is  $\frac{d}{s}$ , which is less than  $\frac{d}{2}$ . Hence, there is a polynomial over  $F_2$  satisfied by  $\zeta$  of degree less than  $d$ , which is impossible.

Now, if  $d$  is even, then by last lemma, we obtain a polynomial of degree  $d$  satisfied by  $\zeta$  and  $\zeta^{-1}$ . It implies that there is a polynomial of degree  $d - 1$  satisfied by  $\zeta + \zeta^{-1}$ . Hence  $[F_2(\zeta + \zeta^{-1}) : F_2] < d$  and therefore,  $[F_2(\zeta) : F_2(\zeta + \zeta^{-1})] = 2$ . Further, if  $d$  is odd, then  $[F_2(\zeta) : F_2(\zeta + \zeta^{-1})] \neq 2$ . Hence  $F_2(\zeta) = F_2(\zeta + \zeta^{-1})$  and so  $[F_2(\zeta + \zeta^{-1}) : F_2] = d$ .  $\square$

*Proof of the Theorem.* Let the cyclotomic polynomial

$$\phi_p(x) = f_1(x)f_2(x) \dots f_s(x)$$

be the product of irreducible factors over  $F_2$ , where  $s = \frac{\phi(p)}{d}$ . Assume that  $\gamma_i$  is a root of irreducible factor  $f_i(x)$  over  $F_2$ . Define a matrix representation of  $D_{2p}$ ,

$$T_{\gamma_i} : D_{2p} \rightarrow M_2(F_2(\gamma_i + \gamma_i^{-1}))$$

by the assignment

$$a \mapsto \begin{pmatrix} 0 & 1 \\ 1 & \gamma_i + \gamma_i^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 0 \\ \gamma_i + \gamma_i^{-1} & 1 \end{pmatrix}$$

If  $d$  is even, then define  $T = T_0 \oplus T_{\gamma_1} \oplus T_{\gamma_2} \oplus \dots \oplus T_{\gamma_s}$ , the direct sum of the given representations  $T_{\gamma_i}, 1 \leq i \leq s$ , and  $T_0$  is the trivial representation of  $D_{2p}$  over  $F_2$  of degree 1.

Suppose  $d$  is odd. Lemma (2) implies that  $\gamma_i$  and  $\gamma_i^{-1}$  are roots of the different irreducible factors of  $\phi_p(x)$ . If  $\gamma_i^{-1}$  is a root of  $f_j(x)$ , then choose  $\gamma_j = \gamma_i^{-1}$ . Without loss of generality, assume that  $\gamma_1, \gamma_2, \dots, \gamma_{s'}$  are the roots of distinct irreducible factors of  $\phi_p(x)$  such that  $\gamma_i \neq \gamma_j^{-1}$  for  $1 \leq i, j \leq s'$ . Then define  $T = T_0 \oplus_{i=1}^{s'} T_{\gamma_i}$ , the direct sum of all distinct matrix representation. Therefore,

$$T : D_{2p} \rightarrow \mathcal{U}(F_2 \oplus M_2(F_2(\gamma_1 + \gamma_1^{-1})) \oplus \dots \oplus M_2(F_2(\gamma_k + \gamma_k^{-1})))$$

given by

$$a \mapsto \left( 1, \begin{pmatrix} 0 & 1 \\ 1 & \gamma_1 + \gamma_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ 1 & \gamma_k + \gamma_k^{-1} \end{pmatrix} \right)$$

and

$$b \mapsto \left( 1, \begin{pmatrix} 1 & 0 \\ \gamma_1 + \gamma_1^{-1} & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ \gamma_k + \gamma_k^{-1} & 1 \end{pmatrix} \right)$$

is a group homomorphism, where  $k = s = \frac{\phi(p)}{d}$ , if  $d$  is even and  $k = s' = \frac{\phi(p)}{2d}$  if  $d$  is odd.

Extend this group homomorphism  $T$  to the algebra homomorphism

$$T' : F_2 D_{2p} \rightarrow F_2 \oplus M_2(F_2(\gamma_1 + \gamma_1^{-1})) \oplus \dots \oplus M_2(F_2(\gamma_k + \gamma_k^{-1})),$$

where  $M_2(F_2(\gamma_i + \gamma_i^{-1}))$  is the algebra of  $2 \times 2$  matrices over the field  $F_2(\gamma_i + \gamma_i^{-1})$ .

Note that the representation  $T_{\gamma_i}$  is equivalent to  $S_{\gamma_i}$ , where

$$S_{\gamma_i}(a) = \begin{pmatrix} \gamma_i & 0 \\ 0 & \gamma_i^{-1} \end{pmatrix}, \quad S_{\gamma_i}(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence, for  $x \in D_{2p}$ ,  $T_{\gamma_i}(x) = M_i S_{\gamma_i}(x) M_i^{-1}$ , where  $M_i = \begin{pmatrix} 1 & 1 \\ \gamma_i & \gamma_i^{-1} \end{pmatrix}$ .

Suppose that  $x = \sum_{i=0}^{p-1} \alpha_i a^i + \sum_{i=0}^{p-1} \beta_i a^i b \in \text{Ker} T'$ . Then  $T'(x) = 0$  implies that

$$\sum_{i=0}^{p-1} \alpha_i + \sum_{i=0}^{p-1} \beta_i = 0 \quad (1)$$

and for  $1 \leq j \leq k$ ,  $\gamma_j$  and  $\gamma_j^{-1}$  satisfies the polynomials  $g(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{p-1} x^{p-1}$  and  $h(x) = \beta_0 + \beta_1 x + \cdots + \beta_{p-1} x^{p-1}$  over  $F_2$ . It follows that irreducible factors of  $\phi_p(x)$  are factors of  $g(x)$  and  $h(x)$ . Further, since all factors are co-prime, it follows that  $\phi_p(x)$  divides  $g(x)$  and  $h(x)$  and hence  $\alpha_i = \alpha_j$ , and  $\beta_i = \beta_j$ ,  $0 \leq i, j \leq p-1$ . Thus, from equation (1), we have  $\alpha_i = \beta_i$ ,  $0 \leq i \leq p-1$  and therefore  $\text{Ker} T' = \widehat{F_2 D_{2p}}$ .

Further, the dimension of  $(F_2 D_{2p} / \widehat{F_2 D_{2p}})$  and  $F_2 \bigoplus_{i=1}^k M_2(F_2(\gamma_i + \gamma_i^{-1}))$  over  $F_2$  are same. Hence

$$F_2 D_{2p} / \widehat{F_2 D_{2p}} \cong F_2 \bigoplus_{i=1}^k M_2(F_2(\gamma_i + \gamma_i^{-1})).$$

Since  $\widehat{F_2 D_{2p}}$  is nilpotent,  $T'$  induces an epimorphism

$$T'' : \mathcal{U}(F_2 D_{2p}) \rightarrow \prod_{i=1}^k GL_2(F_2(\gamma_i + \gamma_i^{-1}))$$

such that  $\ker T'' = \langle 1 + \widehat{D_{2p}} \rangle$ . Hence

$$\mathcal{U}(F_2 D_{2p}) / \langle 1 + \widehat{D_{2p}} \rangle \cong \prod_{i=1}^k GL_2(F_2(\gamma_i + \gamma_i^{-1}))$$

and therefore the result follows.

### STRUCTURE OF $\mathcal{B}(F_2 D_{2p})$

**Theorem 4.** *Let  $p$  be an odd prime such that order of 2 mod  $p$  is  $d$ .*

*Then, the group generated by the bicyclic units, i.e.,*

$$\mathcal{B}(F_2 D_{2p}) \cong \begin{cases} \underbrace{SL_2(F_{2^{\frac{d}{2}}}) \times SL_2(F_{2^{\frac{d}{2}}}) \cdots \times SL_2(F_{2^{\frac{d}{2}}})}_{\frac{\phi(p)}{d} \text{ copies}}, & \text{if } d \text{ is even} \\ \underbrace{SL_2(F_{2^d}) \times SL_2(F_{2^d}) \cdots \times SL_2(F_{2^d})}_{\frac{\phi(p)}{2d} \text{ copies}}, & \text{if } d \text{ is odd} \end{cases}$$

where  $SL_2(F)$  is the special linear group of degree 2 over  $F$ .

We need the following lemmas:

**Lemma 5.**  $D_{2p} \cap \mathcal{B}(F_2 D_{2p}) = \langle a \rangle$ .

*Proof.* Since  $D_{2p}$  is in the normalizer of  $\mathcal{B}(F_2 D_{2p})$ , it implies that  $\mathcal{B}(F_2 D_{2p}) \cap D_{2p}$  is a normal subgroup of  $D_{2p}$ . Therefore, it is either a trivial subgroup or  $\langle a \rangle$ .

We claim that  $b \notin D_{2p} \cap \mathcal{B}(F_2 D_{2p})$ . For that we define a map

$$f : D_{2p} \rightarrow \langle g \mid g^2 = 1 \rangle$$

such that

$$f(a^i) = 1 \text{ and } f(a^i b) = g, 0 \leq i \leq p-1.$$



Note that it is a group homomorphism and we can extend this linearly to an algebra homomorphism  $f'$  from  $F_2D_{2p}$  to  $F_2\langle g \rangle$ . It is easy to see that the image of the bicyclic units under  $f'$  is 1. If  $b \in \mathcal{B}(F_2D_{2p})$ , then  $f'(b) = 1$ , which is not possible. Therefore,  $b \notin \mathcal{B}(F_2D_{2p})$ . This shows that  $D_{2p} \cap \mathcal{B}(F_2D_{2p}) \neq D_{2p}$ .

Also observe that

$$u_{ab,a}u_{ab,a^2} \cdots u_{ab,a^l} = ab(1 + \widehat{D_{2p}})$$

$$\text{and } u_{b,a}u_{b,a^2} \cdots u_{b,a^l} = b(1 + \widehat{D_{2p}}),$$

where  $u_{a^j b, a^i} = 1 + (a^i + a^{-i})(1 + a^j b)$  is a bicyclic unit of the group algebra  $F_2D_{2p}$  and  $l = \frac{p-1}{2}$ . It implies that

$$a = u_{ab,a}u_{ab,a^2} \cdots u_{ab,a^l}u_{b,a^l} \cdots u_{b,a^2}u_{b,a}.$$

Hence,  $D_{2p} \cap \mathcal{B}(F_2D_{2p}) = \langle a \rangle$ . □

**Lemma 6.** *For  $1 \leq i \leq k$ , let  $\gamma_i$  be the primitive  $p$ -th root of unity described in the proof of the Theorem 1. Then the minimal polynomials of  $\gamma_i + \gamma_i^{-1}$ ,  $1 \leq i \leq k$ , are distinct.*

*Proof.* Suppose that  $d$  is even. If  $\zeta$  is a primitive  $p$ -th root of unity, then  $[F_2(\zeta + \zeta^{-1}) : F_2] = \frac{d}{2}$ . Assume that

$$f(x) = a_0 + a_1x + \cdots + a_{\frac{d}{2}}x^{\frac{d}{2}}$$

is the minimal polynomial over  $F_2$  satisfied by both  $\gamma_i + \gamma_i^{-1}$  and  $\gamma_j + \gamma_j^{-1}$  for  $i \neq j$ . It implies that there is a polynomial of degree  $d$  over  $F_2$

satisfied by both  $\gamma_i$  and  $\gamma_j$ . This is a contradiction, because the minimal polynomials of  $\gamma_i$  and  $\gamma_j$  over  $F_2$  are co-prime. Hence the result follows. Further, if  $d$  is odd, then  $[F_2(\zeta + \zeta^{-1}) : F_2] = d$ . Let  $f(x)$  be the minimal polynomial over  $F_2$  satisfied by both  $\gamma_i + \gamma_i^{-1}$  and  $\gamma_j + \gamma_j^{-1}$  for  $i \neq j$ . It follows that there is a polynomial  $g(x)$  of degree  $2d$  over  $F_2$  satisfied by  $\gamma_i, \gamma_j, \gamma_j$  and  $\gamma_j^{-1}$ . Since  $d$  is odd, the minimal polynomials of  $\gamma_i^{\pm 1}$  and  $\gamma_j^{\pm 1}$  over  $F_2$  are co-prime. Hence the product of the minimal polynomials divides  $g(x)$ , which is a contradiction. This completes the proof of the lemma.  $\square$

**Proof of the theorem:** Observe that the image of the bicyclic units of the group algebra  $F_2 D_{2p}$  are in  $\prod_{i=1}^k SL_2(F_2(\gamma_i + \gamma_i^{-1}))$  under the map  $T''$ . Suppose  $T'''$  is the restricted map of  $T''$  to  $\mathcal{B}(F_2 D_{2p})$ , i.e.,

$$T''' : \mathcal{B}(F_2 D_{2p}) \rightarrow \prod_{i=1}^k SL_2(F_2(\gamma_i + \gamma_i^{-1}))$$

such that  $T'''(x) = T''(x)$  for  $x \in \mathcal{B}(F_2 D_{2p})$ . Then  $\ker T''' \leq \ker T''$ . Since  $b \notin \mathcal{B}(F_2 D_{2p})$  and  $u_{b,a} u_{b,a^2} \cdots u_{b,a^l} = b(1 + \widehat{D_{2p}})$ , it follows that  $\ker T''' = \{1\}$ .

Further, it is known that

$$SL_2(F_2(\zeta + \zeta^{-1})) = \left\langle \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \mid u, v \in F_2(\zeta + \zeta^{-1}) \right\rangle.$$

To prove  $T'''$  is onto, it is sufficient to prove that the elements of the

form

$$\left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ u_i & 1 \end{pmatrix}, \dots \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

and

$$\left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix}, \dots \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

where  $u_i, v_i \in F_2(\gamma_i + \gamma_i^{-1})$  have a preimage in  $\mathcal{B}(F_2D_{2p})$  under  $T'''$  for all  $1 \leq i \leq k$ .

Assume that  $y_i = \prod_{\substack{j=1 \\ j \neq i}}^k f'_j(\gamma_i + \gamma_i^{-1})$ , such that  $f'_j(x)$  is the minimal

polynomial of  $\gamma_j + \gamma_j^{-1}$  over  $F_2$ . If  $g(x) = \prod_{\substack{j=1 \\ j \neq i}}^k f'_j(x)$ , then  $g(\gamma_j + \gamma_j^{-1}) = 0$  for  $1 \leq j \leq k, j \neq i$  and  $g(\gamma_i + \gamma_i^{-1}) = y_i$ , a nonzero element of  $F_2(\gamma_i + \gamma_i^{-1})$ .

Take  $\{y_i, y_i(\gamma_i + \gamma_i^{-1}), \dots, y_i(\gamma_i + \gamma_i^{-1})^{t-1}\}$  as a basis of  $F_2(\gamma_i + \gamma_i^{-1})$  over  $F_2$ , where  $t = [F_2(\gamma_i + \gamma_i^{-1}) : F_2]$ . Therefore, any element  $u_i$  of  $F_2(\gamma_i + \gamma_i^{-1})$  can be written as  $u_i = y_i \sum_{j=0}^{t-1} \alpha_j(\gamma_i + \gamma_i^{-1})^j$ . Assume that  $u'(x) = g(x)u(x)$ , where  $u(x) = \alpha_0 + \alpha_1x + \dots + \alpha_{t-1}x^{t-1} \in F_2[x]$  and therefore  $u'(x) \in F_2[x]$ . It is clear that  $u'(\gamma_i + \gamma_i^{-1}) = u_i$  and  $u'(\gamma_j + \gamma_j^{-1}) = 0$  for  $1 \leq j \leq k, j \neq i$ . Further, if  $u'(x) = a_0 + a_1x + \dots + a_mx^m$ , then the generator  $X_i$ , whose  $i$ -th component is  $\begin{pmatrix} 1 & 0 \\ u_i & 1 \end{pmatrix}$  and other components are the identity matrix, can be written as  $X_i = e_0e_1 \dots e_m$ . Here  $e_j$  is an element of  $\prod_{i=1}^k SL_2(F_2(\gamma_i + \gamma_i^{-1}))$  such that

the  $r$ -th component of  $e_j$  is  $\begin{pmatrix} 1 & 0 \\ a_j(\gamma_r + \gamma_r^{-1})^j & 1 \end{pmatrix}$ , for  $0 \leq j \leq m$ ,  $1 \leq r \leq k$ . Now we will prove that the preimage of  $e_j$  is in  $\mathcal{B}(F_2 D_{2p})$  under the map  $T'''$ .

If  $a_j = 0$ , then it is trivial. Now assume that  $a_j = 1$ . Suppose that  $M = (M_1, M_2, \dots, M_k)$ , where  $M_r = \begin{pmatrix} 1 & 1 \\ \gamma_r & \gamma_r^{-1} \end{pmatrix}$ . If

$$(\gamma_r + \gamma_r^{-1})^{j-1} = b_0 + \sum_{s=1}^{l-1} b_s(\gamma_r^s + \gamma_r^{-s}),$$

where  $b_i \in F_2$  then the  $r$ -th component of  $M^{-1}e_jM$  is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (b_0 + \sum_{s=1}^{l-1} b_s(\gamma_r^s + \gamma_r^{-s})) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

By extending the matrix representation  $S_{\gamma_r}$  to an algebra homomorphism over  $F_2$ , we obtain that this element is an image of  $\alpha$  under the algebra homomorphism  $S_{\gamma_r}$ , where  $\alpha = 1 + (b_0 + \sum_{s=1}^{l-1} b_s(a^s + a^{-s}))(1 + b)$ . Since  $S_{\gamma_r}(\alpha) = M_r^{-1}T_{\gamma_r}(\alpha)M_r$ , it follows that  $e_j = T''(\alpha)$ . If  $b_0 = 0$ , then

$$\alpha = \prod_{s=1}^{l-1} (1 + b_s(a^s + a^{-s})(1 + b)),$$

is product of bicyclic units of the group algebra  $F_2 D_{2p}$ . Now if  $b_0 = 1$ , then

$$\alpha = b \prod_{s=1}^{l-1} (1 + b_s(a^s + a^{-s})(1 + b)).$$

Since  $b(1 + \widehat{D_{2p}}) = u_{b,a} \dots u_{b,a^l}$  and  $1 + \widehat{D_{2p}}$  is in the kernel of  $T''$ , it implies that  $\alpha(1 + \widehat{D_{2p}})$  is the preimage of  $e_j$  under the map  $T'''$ .

Therefore, the preimage of  $X_i$  is in  $\mathcal{B}(F_2D_{2p})$ . Similarly we can prove the same thing for other generators. Then

$$\prod_{i=1}^k SL_2(F_2(\gamma_i + \gamma_i^{-1})) = T'''(\mathcal{B}(F_2D_{2p})).$$

Hence

$$\mathcal{B}(F_2D_{2p}) \cong \prod_{i=1}^k SL_2(F_2(\gamma_i + \gamma_i^{-1}))$$

and so

$$|\mathcal{B}(F_2D_{2p})| = \begin{cases} (2^{\frac{d}{2}}(2^d - 1))^k & \text{if } d \text{ is even} \\ (2^d(2^{2d} - 1))^k & \text{if } d \text{ is odd.} \end{cases}$$

## THE STRUCTURE OF UNITARY SUBGROUP AND UNIT GROUP

**Theorem 7.** *The unitary subgroup  $\mathcal{U}_*(F_2D_{2p})$  of the group algebra  $F_2D_{2p}$  is the semidirect product of the normal subgroup  $\mathcal{B}(F_2D_{2p})$  with the group  $\langle b \rangle$ . Further,  $\mathcal{U}(F_2D_{2p}) = \mathcal{U}_*(F_2D_{2p}) \times \prod_{i=1}^k \langle z_i \rangle$ , where  $z_i$  is an invertible element in the center of the group algebra  $F_2D_{2p}$  of order  $2^{\frac{d}{2}} - 1$ , if  $d$  is even; otherwise it is of order  $2^d - 1$ .*

*Proof.* Since  $GL_2(F_2(\gamma + \gamma^{-1}))$  is the direct product of  $SL_2(F_2(\gamma + \gamma^{-1}))$  with the group consisting of all nonzero scalar matrices, we have

$$\mathcal{U}(F_2D_{2p})/V \cong \prod_{i=1}^k (SL_2(F_2(\gamma_i + \gamma_i^{-1})) \times (F_2(\gamma_i + \gamma_i^{-1}))^* I_{2 \times 2}),$$

where  $F_2(\gamma + \gamma^{-1})^*$  is the group of all nonzero elements of  $F_2(\gamma + \gamma^{-1})$ .

Let  $F_2(\gamma_i + \gamma_i^{-1})^* = \langle \eta_i \rangle$  for  $1 \leq i \leq k$ . Since  $y_i = \prod_{\substack{j=1 \\ j \neq i}}^k f'_j(\gamma_i + \gamma_i^{-1})$

is a non zero element of  $F_2(\gamma_i + \gamma_i^{-1})$ , take  $\{y_i, y_i(\gamma_i + \gamma_i^{-1}), \dots, y_i(\gamma_i + \gamma_i^{-1})^{t-1}\}$  as a basis of  $F_2(\gamma_i + \gamma_i^{-1})$  over  $F_2$ . Here  $t = \frac{d}{2}$  when  $d$  is even;

otherwise  $t = d$ . Therefore,  $\eta_i = y_i h_i(\gamma_i + \gamma_i^{-1}) = h'_i(\gamma_i + \gamma_i^{-1})$ , where

$h_i(x)$  and  $h'_i(x) \in F_2[x]$ . Also note that  $h'_i(\gamma_j + \gamma_j^{-1}) = 0$  for  $i \neq j$ . If the

constant coefficient of  $h'_i(x)$  is  $\alpha_{\eta_i}$ , then the image of  $h'_i(a + a^{-1})$  under

the map  $T'$  is the element  $x'_i$  such that the first component of  $x'_i$  is  $\alpha_{\eta_i}$ ,

$(i+1)$ -th component is  $\begin{pmatrix} \eta_i & 0 \\ 0 & \eta_i \end{pmatrix}$  and all the remaining components

are zero matrix. Further, if  $y_0(x) = \prod_{i=1}^k f'_i(x)$ , then  $y_0(\gamma_i + \gamma_i^{-1}) =$

$0, 1 \leq i \leq k$  and the constant coefficient of  $y_0(x)$  is 1. It follows that

the image of  $y_0(a + a^{-1})$  under the map  $T'$  is the element whose first

component is 1 and remaining components are zero. Choose  $x_i$  such

that  $(i+1)$ -th component is  $\begin{pmatrix} \eta_i & 0 \\ 0 & \eta_i \end{pmatrix}$  and the remaining components

are identity matrix. If  $z_i$  denotes a preimage of  $x_i$ , then either

$$z_i = \sum_{\substack{j=1 \\ j \neq i}}^k h'_j(a + a^{-1})^{2^t-1} + h'_i(a + a^{-1})$$

or

$$z_i = \sum_{\substack{j=1 \\ j \neq i}}^k h'_j(a + a^{-1})^{2^t-1} + h'_i(a + a^{-1}) + y_0(a + a^{-1})$$

which are in the center of  $\mathcal{U}(F_2 D_{2p})$  of order  $2^t - 1$ . Further, since

$\langle z_i \rangle \cap \langle z_j \mid 1 \leq j \leq k, j \neq i \rangle = \{1\}$ , take  $W = \prod_{i=1}^k \langle z_i \rangle$ . Also, note that

$W \cap \mathcal{U}_*(F_2D_{2p}) = \{1\}$  and therefore  $\mathcal{U}(F_2D_{2p}) = W \times (\mathcal{B}(F_2D_{2p}) \rtimes \langle b \rangle)$  and  $\mathcal{U}_*(F_2D_{2p}) = \mathcal{B}(F_2D_{2p}) \rtimes \langle b \rangle$ .  $\square$

**Corollary 8.** *The group generated by bicyclic units  $\mathcal{B}(F_2D_{2p})$  and the unitary subgroup  $\mathcal{U}_*(F_2D_{2p})$  are normal subgroups of  $\mathcal{U}(F_2D_{2p})$ .*

**Corollary 9.** *The commutator subgroup  $\mathcal{U}'(F_2D_{2p}) = \mathcal{U}'_*(F_2D_{2p})$ . Also,  $\mathcal{U}'(F_2D_{2p})$  is a normal subgroup of  $\mathcal{B}(F_2D_{2p})$ .*

*Proof.* Since  $\mathcal{U}(F_2D_{2p}) = W \times \mathcal{U}_*(F_2D_{2p})$  such that  $W$  is in the center of  $F_2D_{2p}$ , it follows that  $\mathcal{U}'(F_2D_{2p}) = \mathcal{U}'_*(F_2D_{2p})$ . Further, since  $\mathcal{U}_*(F_2D_{2p}) = \mathcal{B}(F_2D_{2p}) \rtimes \langle b \rangle$  and  $b$  is in the normalizer of  $\mathcal{B}(F_2D_{2p})$ , it implies that  $\mathcal{U}'_*(F_2D_{2p}) \leq \mathcal{B}(F_2D_{2p}) \leq \mathcal{U}_*(F_2D_{2p})$  and hence the result follows.  $\square$

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